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Triangular differential quadrature and its application to elastostatic analysis of Reissner plates

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Abstract

The recently proposed triangular differential quadrature method (TDQM) is further elaborated in the paper. Explicit formulae to calculate the weighting coefficients for uniform grid are developed. The TDQM is applied to the elastostatic analysis of Reissner plates. In comparison with other available numerical results, good accuracy and rapid convergence are achieved, indicating that the TDQM has attractive potential as a novel numerical technique.

Keywords: Differential quadrature method; Triangular differential quadrature method; Weighting coefficients; Reissner plates

1. Introduction

Many physical and engineering problems are described by various differential equations which can be solved using various numerical methods, such as the finite element method (FEM), finite difference method (FDM) and the boundary element method (BEM), etc. These numerical methods have been shown to be the most powerful numerical tools and able to deal with various practical problems. There have been many commercial computer codes available. However, these methods are not always the most efficient since a large number of degrees of freedoms are often required even to gain a moderate accuracy and systematic knowledge background is needed for an analyst. As an alternative numerical approach to solve differential equations, the differential quadrature method (DQM) has been studied for years since it was first introduced by Bellman and Casti (1971). It has been shown by many researchers that the DQM is an attractive numerical method with high efficiency and accuracy. The conventional DQM is mostly effective for one-dimensional problems and multi-dimensional problems with geometrically regular domain. To deal with problems on irregular geometric domains, transformation has to be conducted to map a non-rectangular physical domain into a normalized computational domain (Lam, 1993; Bert and Malik, 1996). As a result, a simple governing equation is often transformed into a lengthy and complicated one especially for high order differential equations. In addition, as pointed out by Zhong (1998), singularity arises in differential

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quadrature analysis for problems on a triangular domain. In the implementation of DQM on a triangular domain, one edge of the mesh has to degenerate into a single point, resulting in over-dense grid near the point and therefore the unnecessary high computational cost.

Most recently, a triangular differential quadrature method (TDQM) was proposed by Zhong (2000), which is able to overcome the above difficulty in the analysis of problems on triangular domains. The consequent avoidance of transformation and mathematical convenience make it become a promising new numerical tool to deal with multidimensional problems. More significant is its potential to be imbued with great feasibility of triangles in domain decomposition technique. In addition, the generalization of the present philosophy will derive a series of useful new techniques in three-dimensional analysis.

In this paper, the TDQM is discussed in detail. Explicit formulae to calculate the weighting coefficients are provided based on generalized Lagrangian interpolation on triangular domain. To further demonstrate the TDQM, elastostatic analysis of Reissner plates is conducted. In comparison with other numerical method, rapid convergence and good to excellent agreement are achieved with rather less grid points. It is concluded that the TDQM is a very promising numerical tool.

2. Triangular differential quadrature method

As proposed by Zhong (2000), the triangular domain is first discretized into a uniform grid system. In the system, a vertex identifies the opposite edge and the normal to the edge identifies the corresponding direction, i.e., vertex 1 opposite edge 1 and direction 1 normal to edge 1. Parallel lines are drawn which divide the distance between vertex 1 and edge 1 into m equal segments in direction 1. Each line is identified with a digit from 0 to m, the line 0 being coincident with edge 1 and line m passing through vertex 1. A typical line is denoted by p in direction 1. Same procedures are repeated in the other two directions, respectively. The typical lines in direction 2 (normal to edge 2) and direction 3 (normal to edge 3) are designated as q and r, respectively. Apparently, a typical point in the mesh is identified by three digits p, q, r, consistent with the designation of typical lines in the three directions. The area coordinates for the typical point are p/m, q/m, r/m. It is noted that

$$p+q+r=m, \quad 0 \leqslant p,q,r \leqslant m. \tag{1}$$

Altogether, there are

$$M=(m+1)(m+2)/2 \tag{2}$$

grid points generated in the entire triangular domain. A pictorial description of the grid system is given in Fig. 1.

As suggested by Zhong (2000), in the TDQM, a partial derivative of a function with respect to a space variable at a grid point is approximated by the weighted linear summation of function values at all grid points in the entire triangular domain. Hence,

$$D_n\{f(x,y)\}_{\alpha\beta\gamma}=\sum_{j=0}^{m}\sum_{i=0}^{m-j}C^{(n)}_{\alpha\beta\gamma,pqr}f_{pqr}, \tag{3}$$

where D_n is a differential operator of order n; subscripts (α,β,γ) stands for the value of the derivative at grid point (α,β,γ); $C^{(n)}_{\alpha\beta\gamma,pqr}$ are the weighting coefficients related to the function values f_{pqr} at points (p,q,r). The summation indices (p,q,r) in the above equation take the following values in the two summation loops:

$$(p,q,r)=(m-i-j,i,j). \tag{4}$$

Introducing the triangular differential quadrature into the differential equation of a problem, a set of simultaneous algebraic equations with the function values at all grid points as unknown variables is

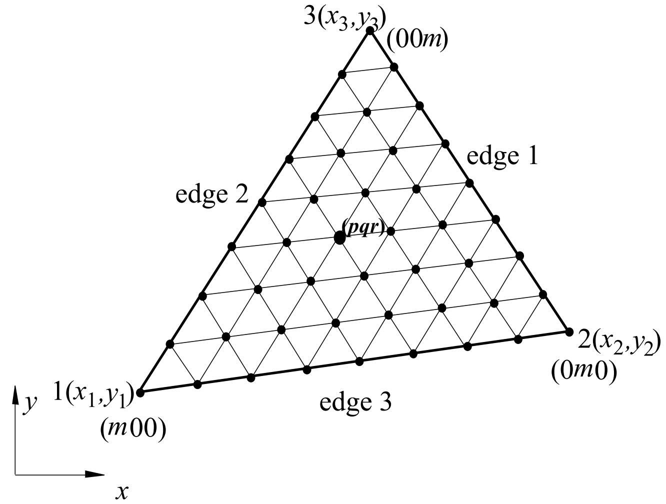

Fig. 1. Grid system in an arbitrary triangular domain.

established. Analogous to the DQM, the governing differential equation is expressed in terms of triangular differential quadrature format at interior grid points of the domain. Meanwhile, the boundary conditions at the edge grid points must be invoked.

One issue, which may be worth addressing, is the terminology of the method. In the paper, the term *quadrature* is inherited despite the actual two-dimensional summation of function values in the present method. On the other hand, a natural extension of the present philosophy in three-dimensional analysis is the introduction of volume coordinates in a triangular pyramid, i.e.

$$D_n\{f(x,y,z)\}_{\alpha\beta\gamma\delta} = \sum_{j=0}^{m}\sum_{i=0}^{m-j}\sum_{k=0}^{m-j-i} C^{(n)}_{\alpha\beta\gamma\delta,pqrs} f_{pqrs}, \tag{5}$$

where the summation indices (p,q,r,s) assume the following values in the three summation loops:

$$(p,q,r,s) = (m-k-i-j,k,i,j). \tag{6}$$

The author is inclined to name it as *tetrahedral* (or *pyramidal*) *quadrature*. If a term *cubature* was given to the present method, one would coin a new term, say, *quartiture* for the method in a tetrahedron. The need for new terms might also arise, for instance, when the present method is combined with the conventional DQM in a prism with triangular cross-section. In this case, *pentahedral quadrature* is eligible following the present nomenclature.

3. Determination and properties of weighting coefficients

One approach to determine the weighting coefficients, as pointed out by Zhong (2000), is to require that Eq. (3) be exact when f takes the following M base functions:

$$f = L_1^p L_2^q L_3^r, \quad 0 \leqslant p,q,r \leqslant m, \tag{7}$$

where the expressions of the three area coordinates of an arbitrary point (x,y) inside a triangular domain can be given as

$$L_i = \frac{1}{2\Delta}(a_i + b_i x + c_i y), \quad i = 1, 2, 3. \tag{8}$$

Δ is the area of the triangle which is expressed in terms of the Cartesian coordinates of the three vertices as

$$2\Delta = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, \tag{9}$$

and the coefficients in Eq. (8) are the values of the determinants of the corresponding cofactor matrices, e.g.

$$a_1 = x_2 y_3 - x_3 y_2, \qquad b_1 = y_2 - y_3, \qquad c_1 = -(x_2 - x_3). \tag{10}$$

The remaining coefficients can be obtained by interchanging subscripts 1, 2, 3. It is noteworthy that the three vertices should be numbered in an anti-clockwise sequence in order to ensure positive value from Eq. (9).

With the chain rule of differentiation, a set of algebraic equations can be established after substituting Eq. (7) for f in Eq. (3). The weighting coefficients can then be determined by solving the resulting simultaneous equations.

Since M grid points are generated in the triangular domain and each grid point has M coefficients, at first sight, there appear $M \times M$ coefficients to be determined for a given derivative. In fact, the weighting coefficients have same interchangeable property as the coefficients in Eq. (8). Suppose that $C^{(x)}_{\alpha\beta\gamma,pqr}$ is known, $C^{(x)}_{\beta\gamma\alpha,qrp}$, $C^{(x)}_{\gamma\alpha\beta,rpq}$, $C^{(x)}_{\alpha\gamma\beta,prq}$, $C^{(x)}_{\beta\alpha\gamma,qpr}$ and $C^{(x)}_{\gamma\beta\alpha,rqp}$ can be obtained by interchanging subscripts 1, 2, 3 associated with b_1, b_2, b_3 in $C^{(x)}_{\alpha\beta\gamma,pqr}$. For instance, in the cubic triangular differential quadrature approximation ($m = 3$)

$$C^{(x)}_{210,300} = (b_1 - 2b_2 - 2b_3)/(6\Delta) = b_1/(2\Delta). \tag{11a}$$

With the above interchanging rule of the triple-subscripts of the weighting coefficients, one can get the following weighting coefficients:

$$C^{(x)}_{021,030} = (b_2 - 2b_3 - 2b_1)/(6\Delta) = b_2/(2\Delta), \tag{11b}$$

$$C^{(x)}_{102,003} = (b_3 - 2b_1 - 2b_2)/(6\Delta) = b_3/(2\Delta), \tag{11c}$$

$$C^{(x)}_{201,300} = (b_1 - 2b_3 - 2b_2)/(6\Delta) = b_1/(2\Delta), \tag{11d}$$

$$C^{(x)}_{120,030} = (b_2 - 2b_1 - 2b_3)/(6\Delta) = b_2/(2\Delta), \tag{11e}$$

$$C^{(x)}_{012,003} = (b_3 - 2b_2 - 2b_1)/(6\Delta) = b_3/(2\Delta). \tag{11f}$$

Thus, the actual minimum number of weighting coefficients to be determined is $m(m+1)$ rather than $M \times M$.

Following the definition of the triangular differential quadrature, it is immediately clear that the weighting coefficients of a higher order derivative can be obtained by means of the self-multiplication of weighting coefficients of first-order derivatives, for instance,

$$C^{(xx)}_{\alpha\beta\gamma,pqr} = \sum_{j=0}^{m}\sum_{i=0}^{m-j} C^{(x)}_{\alpha\beta\gamma,stu} C^{(x)}_{stu,pqr}, \tag{12a}$$

$$C^{(xy)}_{\alpha\beta\gamma,pqr} = \sum_{j=0}^{m}\sum_{i=0}^{m-j} C^{(x)}_{\alpha\beta\gamma,stu} C^{(y)}_{stu,pqr}, \tag{12b}$$

$$C^{(xxxx)}_{\alpha\beta\gamma,pqr} = \sum_{j=0}^{m}\sum_{i=0}^{m-j} C^{(x)}_{\alpha\beta\gamma,stu} C^{(xxx)}_{stu,pqr}. \tag{12c}$$

It follows that the above relations can be written collectively in matrix form as

$$[C^{(xx)}] = [C^{(x)}][C^{(x)}], \tag{13a}$$

$$[C^{(xy)}] = [C^{(x)}][C^{(y)}], \tag{13b}$$

$$[C^{(xxxx)}] = [C^{(x)}][C^{(xxx)}]. \tag{13c}$$

It can be seen that having the weighting coefficient matrices $[C^{(x)}]$ and $[C^{(y)}]$ for first-order derivatives, the weighting coefficients of any higher-order weighting coefficients can be obtained by successive multiplications of $[C^{(x)}]$ and $[C^{(y)}]$.

Analogous to the DQM, it is much more convenient if explicit expression of the weighting coefficients is available. On examination of the grid system, it is easy to find that the generalized Lagrangian interpolation which is commonly used in finite element formulation of shape functions for triangular elements (Gallagher, 1975; Huebner and Thornton, 1982) can be used to form the base functions. Therefore, an alternative approach to determine the weighting coefficients is to require that Eq. (3) be exact when f takes the following M base functions:

$$f_{pqr} = \overline{f}_p(L_1)\overline{f}_q(L_2)\overline{f}_r(L_3), \quad 0 \leqslant p, q, r \leqslant m, \tag{14}$$

where the auxiliary function is given as

$$\overline{f}_p(L_1) = \begin{cases} \prod_{k=1}^{p} \frac{mL_1 - k + 1}{k}, & 1 \leqslant p \leqslant m; \\ 1, & p = 0. \end{cases} \tag{15}$$

Similar expressions for $\overline{f}_q(L_2)$ and $\overline{f}_r(L_3)$ can be defined. It is easy to show that the function in Eq. (14) has the following nature:

$$\begin{aligned} f_{pqr}|_{\alpha\beta\gamma} &= \overline{f}_p(L_1)\Big|_{L_1=\alpha/m} \overline{f}_q(L_2)\big|_{L_2=\beta/m} \overline{f}_r(L_3)\big|_{L_3=\gamma/m} = \delta_{\alpha p}\delta_{\beta q}\delta_{\gamma r} \\ &= \begin{cases} 1, & (\alpha, \beta, \gamma) = (p, q, r); \\ 0, & \text{otherwise.} \end{cases}, \quad 0 \leqslant p, q, r \leqslant m, \end{aligned} \tag{16}$$

where δ is the Kroneckor operator. Take the first derivative of the base function in Eq. (14), one finds

$$\left.\frac{\partial f_{pqr}}{\partial x}\right|_{\alpha\beta\gamma} = \sum_{j=0}^{m}\sum_{i=0}^{m-j} C^{(x)}_{\alpha\beta\gamma,stu} f_{pqr}|_{stu} = \sum_{j=0}^{m}\sum_{i=0}^{m-j} C^{(x)}_{\alpha\beta\gamma,stu}\delta_{ps}\delta_{qt}\delta_{ru} = C^{(x)}_{\alpha\beta\gamma,pqr}. \tag{17}$$

Hence, an explicit expression for the weighting coefficients $C^{(x)}_{\alpha\beta\gamma,pqr}$ is given as follows:

$$C^{(x)}_{\alpha\beta\gamma,pqr} = \begin{bmatrix} \frac{\partial L_1}{\partial x} & \frac{\partial L_2}{\partial x} & \frac{\partial L_3}{\partial x} \end{bmatrix} \begin{Bmatrix} \frac{\partial f_{pqr}}{\partial L_1} \\ \frac{\partial f_{pqr}}{\partial L_2} \\ \frac{\partial f_{pqr}}{\partial L_3} \end{Bmatrix}_{\alpha\beta\gamma} = \begin{bmatrix} \frac{b_1}{2\Delta} & \frac{b_2}{2\Delta} & \frac{b_3}{2\Delta} \end{bmatrix} \begin{Bmatrix} \frac{d\overline{f}_p}{dL_1}\overline{f}_q\overline{f}_r \\ \overline{f}_p\frac{d\overline{f}_q}{dL_2}\overline{f}_r \\ \overline{f}_p\overline{f}_q\frac{d\overline{f}_r}{dL_3} \end{Bmatrix}_{\alpha\beta\gamma}. \tag{18}$$

A similar expression for $C^{(y)}_{\alpha\beta\gamma,pqr}$ can be established, i.e.

$$C^{(y)}_{\alpha\beta\gamma,pqr} = \begin{bmatrix} \frac{\partial L_1}{\partial y} & \frac{\partial L_2}{\partial y} & \frac{\partial L_3}{\partial y} \end{bmatrix} \begin{Bmatrix} \frac{\partial f_{pqr}}{\partial L_1} \\ \frac{\partial f_{pqr}}{\partial L_2} \\ \frac{\partial f_{pqr}}{\partial L_3} \end{Bmatrix}_{\alpha\beta\gamma} = \begin{bmatrix} \frac{c_1}{2\Delta} & \frac{c_2}{2\Delta} & \frac{c_3}{2\Delta} \end{bmatrix} \begin{Bmatrix} \frac{\mathrm{d}\overline{f}_p}{\mathrm{d}L_1}\overline{f}_q\overline{f}_r \\ \overline{f}_p\frac{\mathrm{d}\overline{f}_q}{\mathrm{d}L_2}\overline{f}_r \\ \overline{f}_p\overline{f}_q\frac{\mathrm{d}\overline{f}_r}{\mathrm{d}L_3} \end{Bmatrix}_{\alpha\beta\gamma}. \tag{19}$$

From Eq. (15), the first order derivative of $\overline{f}_p(L_1)$ with respect to L_1 is obtained as

$$\left.\frac{\mathrm{d}\overline{f}_p}{\mathrm{d}L_1}\right|_{L_1=\alpha/m} = \begin{cases} m\sum\limits_{k=1}^{p}\frac{\overline{f}_p(\alpha/m)}{\alpha-k+1}, & 2\leqslant p\leqslant\alpha, \\ \frac{m}{p!}\prod\limits_{\substack{k=1\\k\neq\alpha+1}}^{p}(\alpha-k+1), & 0\leqslant\alpha\leqslant p-1, \\ m, & p=1, \\ 0, & p=0. \end{cases} \tag{20}$$

Similar expressions for the derivatives of $\overline{f}_q(L_2)$ and $\overline{f}_r(L_3)$ with respect to L_2 and L_3 can be derived

$$\left.\frac{\mathrm{d}\overline{f}_q}{\mathrm{d}L_2}\right|_{L_2=\beta/m} = \begin{cases} m\sum\limits_{k=1}^{q}\frac{\overline{f}_q(\beta/m)}{\beta-k+1}, & 2\leqslant q\leqslant\beta, \\ \frac{m}{q!}\prod\limits_{\substack{k=1\\k\neq\beta+1}}^{q}(\beta-k+1), & 0\leqslant\beta\leqslant q-1, \\ m, & q=1, \\ 0, & q=0, \end{cases} \tag{21}$$

$$\left.\frac{\mathrm{d}\overline{f}_r}{\mathrm{d}L_3}\right|_{L_3=\gamma/m} = \begin{cases} m\sum\limits_{k=1}^{r}\frac{\overline{f}_r(\gamma/m)}{\gamma-k+1}, & 2\leqslant r\leqslant\gamma, \\ \frac{m}{r!}\prod\limits_{\substack{k=1\\k\neq\gamma+1}}^{r}(\gamma-k+1), & 0\leqslant\gamma\leqslant r-1, \\ m, & r=1, \\ 0, & r=0. \end{cases} \tag{22}$$

The weighting coefficients for high order derivatives can still be determined by the recurrence relationships provided in Eqs. (12) and (13).

4. Formulation of Reissner plates

The governing equations for a homogeneous and isotropic Reissner plate, in terms of the three displacement components are given as follows:

$$\begin{cases} D\left(\frac{\partial^2\psi_x}{\partial x^2} + \frac{1-v}{2}\frac{\partial^2\psi_x}{\partial y^2} + \frac{1+v}{2}\frac{\partial^2\psi_y}{\partial x\partial y}\right) + G\kappa h\left(\frac{\partial w}{\partial x} - \psi_x\right) = 0, \\ D\left(\frac{\partial^2\psi_y}{\partial y^2} + \frac{1-v}{2}\frac{\partial^2\psi_y}{\partial x^2} + \frac{1+v}{2}\frac{\partial^2\psi_x}{\partial x\partial y}\right) + G\kappa h\left(\frac{\partial w}{\partial y} - \psi_y\right) = 0, \\ G\kappa h\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial\psi_x}{\partial x} - \frac{\partial\psi_y}{\partial y}\right) + q = 0, \end{cases} \tag{23}$$

where w is the deflection, ψ_x and ψ_y are the rotations of the normal against the two coordinate axes; q is the transverse loading intensity applied on the upper surface of the plate; G, h, κ are the shear modulus, plate thickness and shear correction factor which is often taken as $5/6$. D is the flexural rigidity which is given as

$$D = \frac{Eh^3}{12(1 - \nu^2)}, \tag{24}$$

where E, ν are Young's modulus and Poisson's ratio, respectively.

Introducing the triangular differential quadrature, Eq. (23) will be recast into a set of algebraic equations at any grid point (α, β, γ), i.e.

$$\begin{cases} \sum_{j=0}^{m}\sum_{i=0}^{m-j}\Big[\Big(C^{(xx)}_{\alpha\beta\gamma,pqr} + \frac{1-\nu}{2}C^{(yy)}_{\alpha\beta\gamma,pqr}\Big)(\psi_x)_{pqr} + \frac{1+\nu}{2}C^{(xy)}_{\alpha\beta\gamma,pqr}(\psi_y)_{pqr} + \frac{G\kappa h}{D}C^{(x)}_{\alpha\beta\gamma,pqr}w_{pqr}\Big] - \frac{G\kappa h}{D}(\psi_x)_{\alpha\beta\gamma} = 0, \\ \sum_{j=0}^{m}\sum_{i=0}^{m-j}\Big[\Big(C^{(yy)}_{\alpha\beta\gamma,pqr} + \frac{1-\nu}{2}C^{(xx)}_{\alpha\beta\gamma,pqr}\Big)(\psi_y)_{pqr} + \frac{1+\nu}{2}C^{(xy)}_{\alpha\beta\gamma,pqr}(\psi_x)_{pqr} + \frac{G\kappa h}{D}C^{(y)}_{\alpha\beta\gamma,pqr}w_{pqr}\Big] - \frac{G\kappa h}{D}(\psi_y)_{\alpha\beta\gamma} = 0, \\ \frac{G\kappa h}{D}\sum_{j=0}^{m}\sum_{i=0}^{m-j}\Big[\Big(C^{(xx)}_{\alpha\beta\gamma,pqr} + C^{(yy)}_{\alpha\beta\gamma,pqr}\Big)w_{pqr} - C^{(x)}_{\alpha\beta\gamma,pqr}(\psi_x)_{pqr} - C^{(y)}_{\alpha\beta\gamma,pqr}(\psi_y)_{pqr}\Big] + \frac{q}{D} = 0. \end{cases} \tag{25}$$

The bending moments, twisting moments and shear forces and their triangular differential quadrature formats are expressed as

$$(M_x)_{\alpha\beta\gamma} = -D\left(\frac{\partial\psi_x}{\partial x} + \nu\frac{\partial\psi_y}{\partial y}\right)_{\alpha\beta\gamma} = -D\sum_{j=0}^{m}\sum_{i=0}^{m-j}\Big(C^{(x)}_{\alpha\beta\gamma,pqr}(\psi_x)_{pqr} + \nu C^{(y)}_{\alpha\beta\gamma,pqr}(\psi_y)_{pqr}\Big), \tag{26}$$

$$(M_y)_{\alpha\beta\gamma} = -D\left(\frac{\partial\psi_y}{\partial y} + \nu\frac{\partial\psi_x}{\partial x}\right)_{\alpha\beta\gamma} = -D\sum_{j=0}^{m}\sum_{i=0}^{m-j}\Big(C^{(y)}_{\alpha\beta\gamma,pqr}(\psi_y)_{pqr} + \nu C^{(x)}_{\alpha\beta\gamma,pqr}(\psi_x)_{pqr}\Big), \tag{27}$$

$$(M_{xy})_{\alpha\beta\gamma} = -\frac{1-\nu}{2}D\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right)_{\alpha\beta\gamma} = -\frac{1-\nu}{2}D\sum_{j=0}^{m}\sum_{i=0}^{m-j}\Big(C^{(y)}_{\alpha\beta\gamma,pqr}(\psi_x)_{pqr} + \nu C^{(x)}_{\alpha\beta\gamma,pqr}(\psi_y)_{pqr}\Big), \tag{28}$$

$$(Q_x)_{\alpha\beta\gamma} = G\kappa h\left(\frac{\partial w}{\partial x} - \psi_x\right)_{\alpha\beta\gamma} = G\kappa h\sum_{j=0}^{m}\sum_{i=0}^{m-j}\Big(C^{(x)}_{\alpha\beta\gamma,pqr}w_{pqr}\Big) - G\kappa h(\psi_x)_{\alpha\beta\gamma}, \tag{29}$$

$$(Q_y)_{\alpha\beta\gamma} = G\kappa h\left(\frac{\partial w}{\partial y} - \psi_y\right)_{\alpha\beta\gamma} = G\kappa h\sum_{j=0}^{m}\sum_{i=0}^{m-j}\Big(C^{(x)}_{\alpha\beta\gamma,pqr}w_{pqr}\Big) - G\kappa h(\psi_y)_{\alpha\beta\gamma}. \tag{30}$$

Two typical boundary conditions are

(a) *Simply supported edge (S)*

$$\begin{aligned} &w = 0, \qquad M_n = M_x\cos^2\theta + 2M_{xy}\cos\theta\sin\theta + M_y\sin^2\theta = 0, \\ &\psi_s = -(\sin\theta)\psi_x + (\cos\theta)\psi_y = 0, \end{aligned} \tag{31}$$

(b) *Fully-clamped edge (C)*

$$w = 0, \quad \psi_x = 0, \quad \psi_y = 0, \tag{32}$$

where θ is the angle from the x-axis to the outward normal at an edge point.

To conduct triangular differential quadrature analysis of the problem, the differential equations are implemented at the $(m-2)(m-1)/2$ interior grid points, i.e., $1 \leqslant \alpha,\ \beta,\ \gamma \leqslant m-2$; in the meantime, boundary conditions are imposed at the $3m$ grids on the three edges of the triangle. Altogether, there are $3M$ equations with $3M$ unknowns to be solved.

5. Results and discussion

The above triangular differential quadrature procedures are first employed to study an equilateral triangular Reissner plate subjected to uniformly distributed load under different boundary conditions. The plate under SCC boundary conditions is shown in Fig. 2. The problems chosen are aimed to demonstrate the accuracy and convergence speed of the triangular differential quadrature method. In all calculations, Poisson's ratio v is taken as 0.3. In the case when there is no grid pertaining to the centroid, the generalized Lagrangian interpolation in Eq. (14) is adopted to compute the desired displacement components. For example, the deflection at the centroid can be given as

$$w|_{L_1=L_2=L_3=1/3} = \sum_{j=0}^{m}\sum_{i=0}^{m-j} \overline{f}_p(L_1)\big|_{L_1=1/3}\overline{f}_q(L_2)\big|_{L_2=1/3}\overline{f}_r(L_3)\big|_{L_3=1/3} w_{pqr}, \quad 0 \leqslant p,q,r \leqslant m. \tag{33}$$

Meanwhile, the derivatives at the centroid which are needed in the expressions of internal forces can be worked out as well based on Eq. (14), such as

$$\left.\frac{\partial \psi_x}{\partial x}\right|_{L_1=L_2=L_3=1/3} = \sum_{j=0}^{m}\sum_{i=0}^{m-j} (\psi_x)_{pqr} \begin{bmatrix} \frac{b_1}{2\Delta} & \frac{b_2}{2\Delta} & \frac{b_3}{2\Delta} \end{bmatrix} \left\{ \begin{array}{c} \frac{d\overline{f}_p}{dL_1}\overline{f}_q\overline{f}_r \\ \overline{f}_p\frac{d\overline{f}_q}{dL_2}\overline{f}_r \\ \overline{f}_p\overline{f}_q\frac{d\overline{f}_r}{dL_3} \end{array} \right\}_{L_1=L_2=L_3=1/3}. \tag{34}$$

Actually, the displacement components and their derivatives at any point of the triangle can be obtained with reference to the above two equations. The computed deflection and bending moments at the centroid for plates with three different boundary conditions SSS, CCC and SCC are listed in Tables 1–3. In these tables, the following non-dimensional quantities are introduced to represent the deflection and two bending moments at the centroid of the equilateral triangular plate:

$$\overline{w} = 10^2\frac{wEh^3}{qa^4}, \quad \overline{M}_x = 10^2\frac{M_x}{qa^2}, \quad \overline{M}_y = 10^2\frac{M_y}{qa^2}, \tag{35}$$

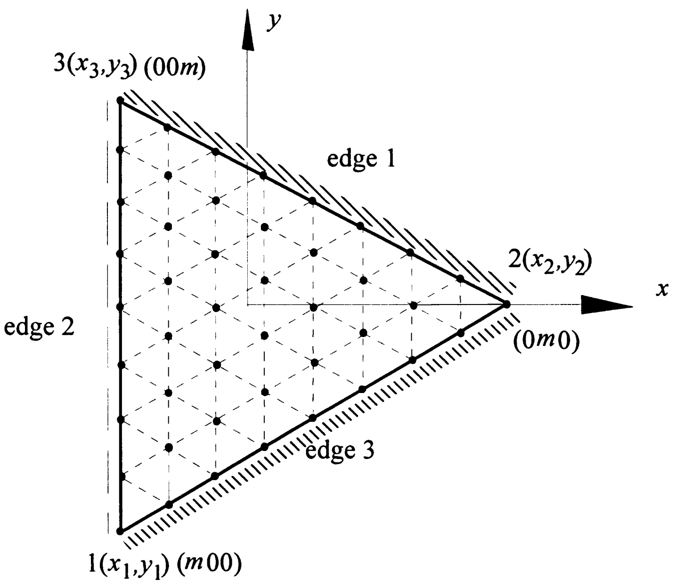

Fig. 2. Grid of equilateral triangular plate under SCC boundary condition.

Table 1
Convergence of TDQ analysis of an equilateral triangle plate under SSS

h/a	m	M	$\overline{w}$	$\overline{M}_x$	$\overline{M}_y$
0.001	3	10	0.8667×10^{-5}	0	0
	4	15	0.6066×10^{-4}	1.3205×10^{-4}	1.3205×10^{-4}
	5	21	0.6319	1.8056	1.8056
	6	28	0.6319	1.8056	1.8056
	Exact		0.6319	1.8056	1.8056
0.01	5	21	0.6328	1.8056	1.8056
	6	28	0.6328	1.8056	1.8056
	FEM[a]		0.6327	1.801	1.808
	Exact		0.6328	1.8056	1.8056
0.1	5	21	0.7186	1.8056	1.8056
	6	28	0.7186	1.8056	1.8056
	Exact		0.7186	1.8056	1.8056
	FEM[a]		0.7186	1.808	1.808
0.2	5	21	0.9786	1.8056	1.8056
	6	28	0.9786	1.8056	1.8056
	Exact		0.9786	1.8056	1.8056
	FEM[a]		0.9786	1.808	1.808

[a] Liu and Liew (1998).

Table 2
Convergence of TDQ analysis of an equilateral triangle plate under CCC

h/a	m	M	$\overline{w}$	$\overline{M}_x$	$\overline{M}_y$
0.01	6	28	0.2012	0.8594	0.8594
	7	36	0.1977	0.8481	0.8481
	11	78	0.1844	0.8190	0.8188
	12	91	0.1841	0.8180	0.8176
	13	105	0.1840	0.8178	0.8178
	14	120	0.1842	0.8182	0.8182
	15	136	0.1840	0.8178	0.8178
	16	153	0.1841	0.8181	0.8181
0.1	7	36	0.2779	0.8389	0.8389
	8	45	0.2785	0.8390	0.8390
	9	55	0.2784	0.8401	0.8401
	10	66	0.2786	0.8403	0.8403
	11	78	0.2785	0.8400	0.8400
	12	91	0.2786	0.8405	0.8405
	13	105	0.2786	0.8403	0.8403
	FEM[a]		0.2787	0.8432	0.8428
0.25	7	36	0.7469	0.8729	0.8729
	8	45	0.7470	0.8752	0.8752
	9	55	0.7470	0.8746	0.8746
	10	66	0.7470	0.8754	0.8754
	11	78	0.7471	0.8751	0.8751
	FEM[a]		0.7471	0.8779	0.8775

[a] Liu and Liew (1998).

Table 3
Convergence of TDQ analysis of an equilateral triangle plate under SCC

h/a	m	M	$\overline{w}$	$\overline{M}_x$	$\overline{M}_y$
	7	36	0.2567	0.8673	1.0955
0.01	11	78	0.2587	0.9057	1.0723
	12	91	0.2584	0.9052	1.0635
	13	105	0.2584	0.9054	1.0629
	14	120	0.2585	0.9052	1.0637
	15	136	0.2584	0.9055	1.0630
0.1	6	28	0.3677	1.0030	1.0866
	7	36	0.3625	0.9873	1.0596
	8	45	0.3620	0.9870	1.0600
	9	55	0.3625	0.9883	1.0614
	10	66	0.3624	0.9883	1.0619
	11	78	0.3622	0.9879	1.0612
	12	91	0.3625	0.9885	1.0622
	13	105	0.3623	0.9881	1.0614
	FEM[a]		0.3624	0.990	1.065
0.25	5	21	0.8792	1.2051	1.0436
	6	28	0.8649	1.2169	1.0564
	7	36	0.8694	1.2179	1.0558
	8	45	0.8676	1.2167	1.0563
	9	55	0.8683	1.2172	1.0574
	10	66	0.8680	1.2180	1.0572
	11	78	0.8682	1.2174	1.0575
	12	91	0.8681	1.2178	1.0574
	13	105	0.8682	1.2176	1.0576
	14	120	0.8681	1.2177	1.0575
	FEM[a]		0.8682	1.220	1.060

[a] Liu and Liew (1998).

where a is the edge length of the equilateral triangle. All FEM solutions are cited from the results of Liu and Liew (1998) using ANSYS computer code with 2116 grid points.

The results for some low order usable grids are listed with the objective to display the convergence threshold. It can be seen that the triangular differential quadrature analysis for equilateral triangular plates under the three different boundary conditions start to converge from $m = 5$ or $m = 6$ with total grid points $M = 21$ or $M = 28$. For equilateral triangular plates with SSS boundary conditions, a noteworthy feature is that the results attained from TDQ analysis with 21 grid points ($m = 5$) are in excellent agreement with available solutions from other sources regardless of the thickness-to-edge–length ratio. An acceptable interpretation comes from the fact that the exact solution of the deflection for thin plates is given in terms of a quintic polynomial (Timoshenko and Woinowsky-Krieger, 1970). The exact solution of moderately thick triangular plates with all three edges simply-supported is also a quintic polynomial (Hu, 1981), i.e.

$$w^{(R)} = w^{(K)} - \frac{h^2}{6\kappa(1-\nu^2)}\left(\frac{\partial^2 w^{(K)}}{\partial x^2} + \frac{\partial^2 w^{(K)}}{\partial y^2}\right), \tag{36}$$

where the superscripts R and K represent the solutions for Reissner plate and Kirchhoff plate, respectively. In contrast, as reported by Liu and Liew (1998), 2116 grid points were used in the finite element code ANSYS analysis to achieve comparable accuracy. For equilateral triangular plates with CCC and SCC boundary conditions, the triangular differential analysis still exhibits good convergent behavior. Three effective decimal digits are stabilized when the number of total grid points is increased to 91 for very thin plate, still being in sharp contrast to the 2116 grid points of finite element analysis. With the increase of plate thickness, it is found that the results converge more rapidly. For thick plate $h/a = 0.25$, the three digits of results are unaltered with mere 36 grid points.

To further demonstrate the applicability of the TDQM, an isosceles right triangular plate under uniform distributed load is also studied (Fig. 3). The exact solution for thin plate is given in terms of double trigonometric series (Timoshenko and Woinowsky-Krieger, 1970)

$$w^{(\mathrm{K})} = \frac{16qa^4}{\pi^6 D}\left[\sum_{m=1,3,5,\ldots}^{\infty}\sum_{n=2,4,6,\ldots}^{\infty}\frac{n\sin m\pi x/a\sin n\pi y/a}{m(n^2-m^2)(m^2+n^2)^2} + \sum_{m=2,4,6,\ldots}^{\infty}\sum_{n=1,3,5,\ldots}^{\infty}\frac{m\sin m\pi x/a\sin n\pi y/a}{n(m^2-n^2)(m^2+n^2)^2}\right]. \tag{37}$$

The computed deflection and two bending moments at the centroid are listed in Table 4. Although the convergence of the bending moments for thin plates is not as satisfactory as that of the deflection, the results

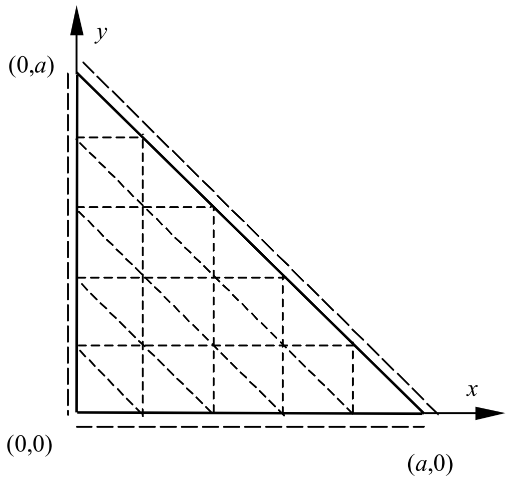

Fig. 3. Simply supported isosceles right triangular plate.

Table 4
Results of isosceles right triangle with three edges simply supported

h/a	m	$\overline{w}$	$\overline{M}_x$	$\overline{M}_y$
0.01	15	0.7015	2.0086	1.7519
	17	0.7020	1.7297	2.0330
	19	0.7022	1.9758	1.7878
	21	0.7024	1.7728	1.9914
	Exact	0.7016[K a]	1.8826	1.8826
	Exact	0.7023[R a]	1.8826	1.8826
0.05	15	0.7237	1.8379	1.9249
	17	0.7239	1.8774	1.8860
	19	0.7239	1.8757	1.8882
	Exact	0.7190	1.8826	1.8826
0.1	15	0.7914	1.9064	1.8562
	17	0.7916	1.8882	1.8752
	19	0.7917	1.8823	1.8816
	Exact	0.7711	1.8826	1.8826

[a] K and R represent the exact solutions based on Kirchhoff theory and Reissner theory, respectively.

are still acceptable. It is believed that the solution is more or less affected by the partial implementation of the simply-supported boundary conditions at the corner. It should be mentioned that there seems to be a significant calculating error in the results of Timoshenko and Woinowsky-Krieger (1970), the theoretical results therefore are obtained based on Eq. (37) rather than using the data provided in their book.

6. Concluding remarks

The newly proposed TDQM has been elaborated in the present study. Explicit expressions to calculate the weighting coefficients were developed. The rapid convergence and satisfactory accuracy of the TDQM has been demonstrated in the present work. In comparison with other available data on static flexural analysis of triangular Reissner plate, good agreement has been reached with quite less grid points. It is concluded that the TDQM is a promising numerical tool in dealing with multi-variable problems. It is believed that the method will gain strong vitality when the domain decomposition technique is incorporated.

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